Directed networks via Persistent Hochschild Homology

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Background

Network data in many guises in life sciences:

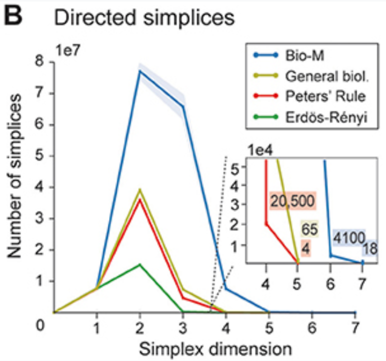
- protein–protein interaction networks
- gene regulatory networks
- vascular network
- neuronal networks

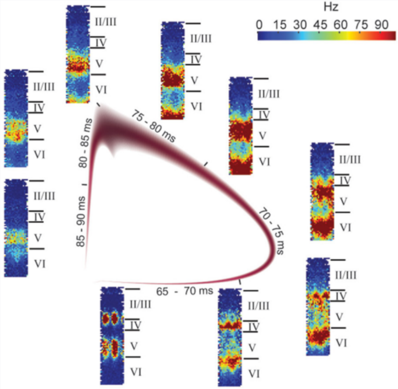
From TDA perspective, what is an appropriate (persistent) homology theory for networks, or directed graphs?

- use homological information to understand some aspects of, for example, network dynamics

Some topological observations in neuronal networks

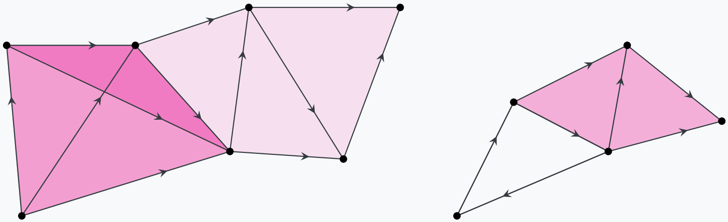
(M. Reimann et al., *Cliques of Neurons Bound into Cavities Provide a Missing Link between Structure and Function*, 2017)



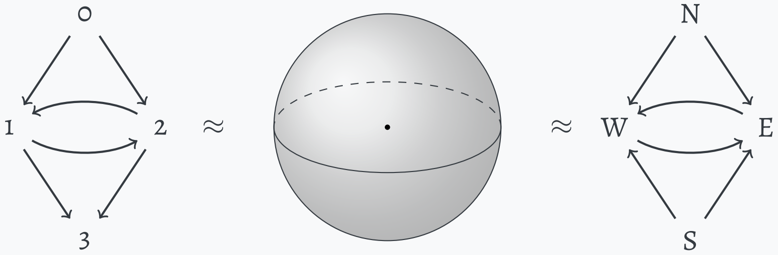


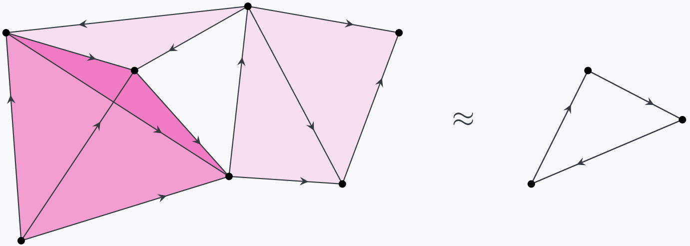
Directed flag complex of a digraph G

n-simplices are totally ordered sets of vertices $v_0 < v_1 < \cdots < v_n$, such that for any $v_i < v_j$, the pair (v_i, v_j) is a directed edge in G.



Topology of directed flag complexes



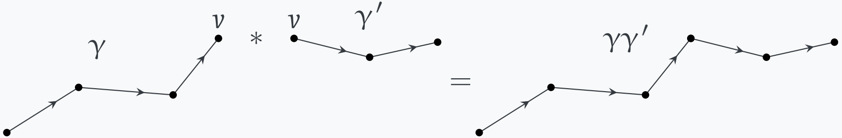


Homology from directed paths?

From network analysis perspective, phenomena travel over the directed edges and paths. *We want homological tools to account for this.*

The **path algebra** kG over a field k associated to the digraph G

- has as a k-vector space a basis consisting of all possible paths γ in G
- and the multiplication is concatenation of paths end-to-end if possible, and 0 otherwise.



Path algebra and Hochschild homology

We can use **Hochschild homology** of associative algebras as a homology theory for directed graphs.

$$\cdots \xrightarrow{b_{n+1}} kG^{\otimes n+1} \xrightarrow{b_n} kG^{\otimes n} \xrightarrow{b_{n-1}} \cdots \xrightarrow{b_1} kG.$$

Happel's combinatorial formula for Betti numbers when G is a connected, acyclic, directed graph:

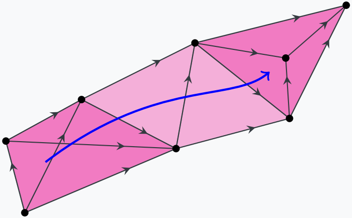
$$\dim_k HH_i(kG) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > 1 \\ 1 - n + \sum_{e \in G} \dim_k s_e kG t_e & \text{if } i = 1, \end{cases}$$

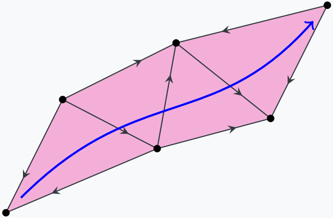
where $n =$ is the number of vertices of G and $s_e kG t_e$ is the subspace generated by all the paths from start of e to end of e. *No higher degree homological information.*

Higher Hochschild homology of digraphs

We introduce higher degree HH by lifting to paths of simplices.

A **connectivity structure** of a digraph $E\colon \mathrm{Sub}(G) \times \mathrm{Sub}(G) \to \{0, 1\}$.





for simplices σ and τ
and directions (d_i, d_j),
$(\sigma, \tau) \mapsto 1$ if either $\sigma \hookrightarrow \tau$
or $d_i(\sigma) \hookleftarrow \alpha \hookrightarrow d_j(\tau)$,
for some q-simplex α

for n-simplices σ and τ,
$(\sigma, \tau) \mapsto 1$ if there is
an $(n-1)$-simplex α and some
$i, j \in \{0, \ldots, n\}$ such that
$d_i(\sigma) = \alpha = d_j(\tau)$, with $i < j$

Persistent Hochschild homology from connectivity digraphs

1. For a digraph G, apply connectivity structure E to get a map

$$\mathbf{Digraph} \xrightarrow{E} \mathbf{Digraph}(\mathbf{of\ simplices})$$

2. Go to the path algebra

$$\mathbf{Digraph} \xrightarrow{E} \mathbf{Digraph}(\mathbf{of\ simplices}) \xrightarrow{k-} k\text{-}\mathbf{Alg}$$

3. Compute Hochschild homology of the path algebra

$$\mathbf{Digraph} \xrightarrow{E} \mathbf{Digraph}(\mathbf{of\ simplices}) \xrightarrow{k-} k\text{-}\mathbf{Alg} \xrightarrow{HH} \mathbf{FinVect}$$

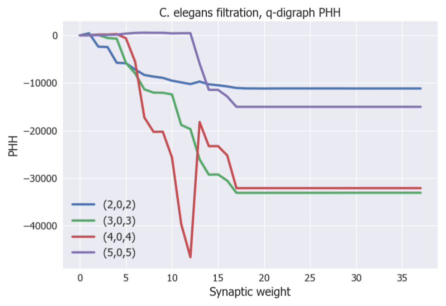
4. Adding filtration and (non-functorial) condensation

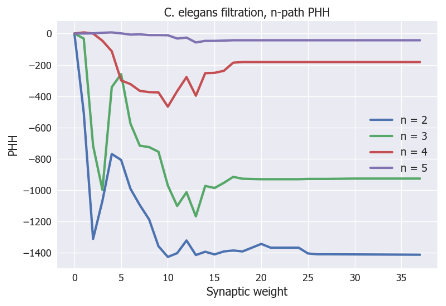
$$(\mathbf{R}, \leqslant) \xrightarrow{\mathcal{F}} \mathbf{Digraph} \xrightarrow{E} \mathbf{Digraph} \xrightarrow{c} \mathbf{Digraph}_{\mathrm{acyc}} \xrightarrow{k-} k\text{-}\mathbf{Alg} \xrightarrow{HH} \mathbf{FinVect}$$

When the pipeline is fully functorial, we have a stability theorem.

Persistent *HH* of C. elegans neuronal network

Filtration induced by the synaptic strength between neurons. The erratic behaviour is due to the condensation operation.





Fin.

Thanks!

Henri Riihimäki, *Simplicial q-connectivity of directed graphs with applications to network analysis*, to appear in SIAM Journal on Mathematics of Data Science, arXiv:2202.07307, 2022.

Luigi Caputi, Henri Riihimäki, *Hochschild homology, and a persistent approach via connectivity digraphs*, Journal of Applied and Computational Topology, 2023.

L. Caputi and H. Riihimäki, *On reachability categories and commuting algebras of quivers*, arXiv:2306.15388, 2023.